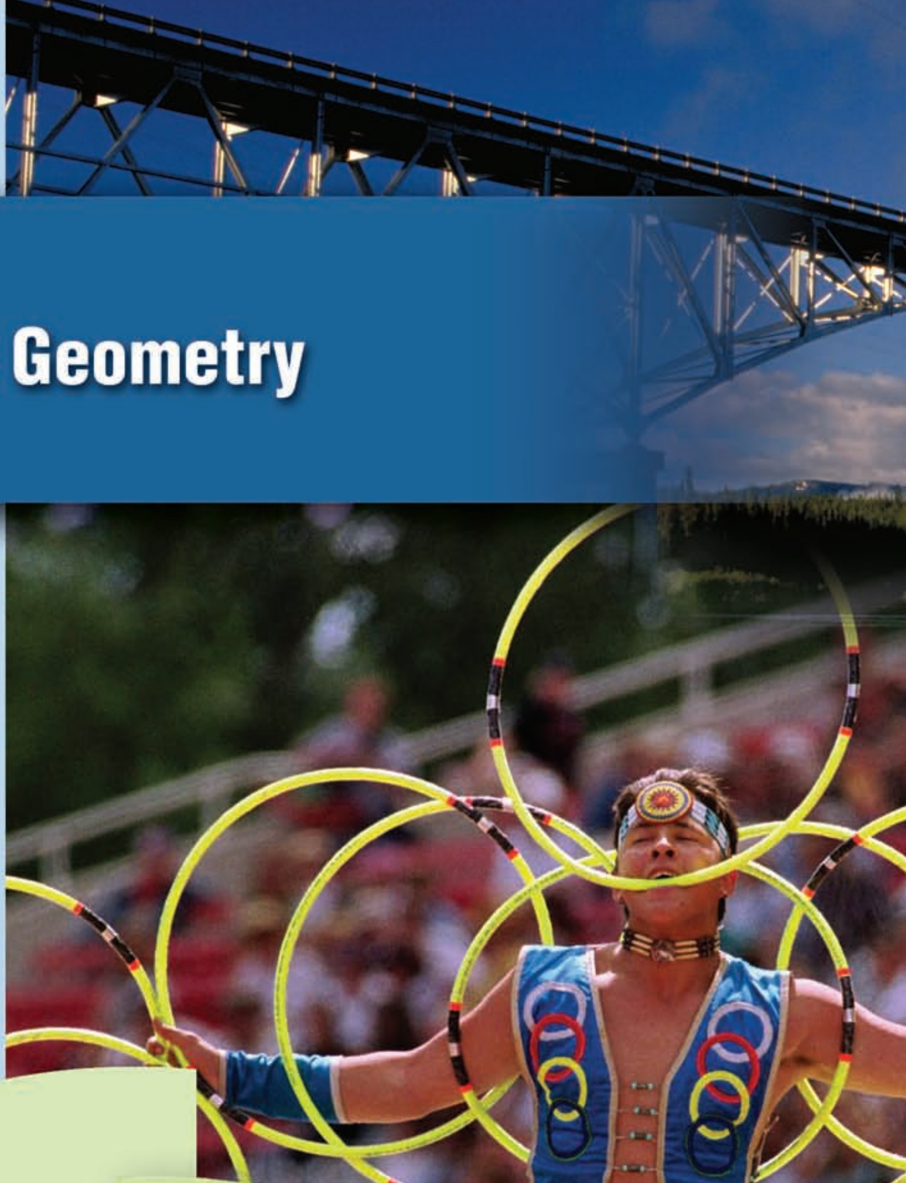
UNIT 8

Circle Geometry



We see circles in nature and in design.
What do you already know about circles?

What You'll Learn

Circle properties that relate:

- a tangent to a circle and the radius of the circle
- a chord in a circle, its perpendicular bisector, and the centre of the circle
- the measures of angles in circles

Why It's Important

Knowing the properties of circles and the lines that intersect them helps us to use circles in designs, to calculate measurements involving circles, and to understand natural objects that are circular.

Key Words

- tangent
- point of tangency
- chord
- arc
- minor arc
- major arc
- central angle
- inscribed angle
- subtended
- inscribed polygon
- supplementary angles

8.1 Properties of Tangents to a Circle

FOCUS

- Discover the relationship between a tangent and a radius, then solve related problems.

This wheel is rolling in a straight line on a flat surface.
The wheel touches the ground at only one point.
Visualize the red spoke extended to the ground.
What angle does the spoke appear to make with the ground?

Investigate

You will need a compass, protractor, and ruler.

➤ Construct a large circle and draw a radius.
Draw a line that touches the circle only at the endpoint of the radius.
Measure the angle between the radius and the line.

➤ Repeat the previous step for other radii and lines that touch the circle at an endpoint of a radius. Record your results.

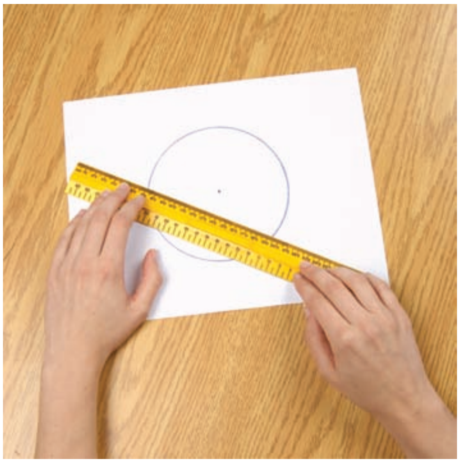
Repeat the procedure for other circles.
Write a statement about what you observe.

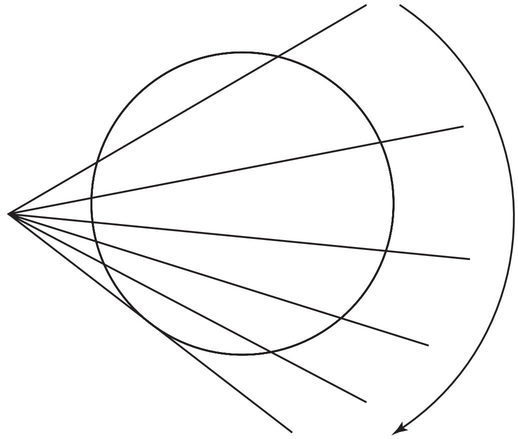
Reflect & Share

Compare your results with those of another pair of students.
What is the mean value of the angles you measured?
What do you think is the measure of the angle between the line you drew and the radius of each circle?

Connect

Imagine fixing one end of a ruler and rotating the ruler across a circle.

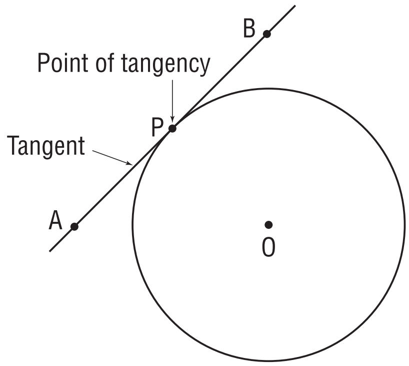




As one edge of the ruler sweeps across the circle, it intersects the circle at 2 points.
Just as the ruler leaves the circle, it intersects the circle at 1 point.
The edge of the ruler is then a tangent to the circle.

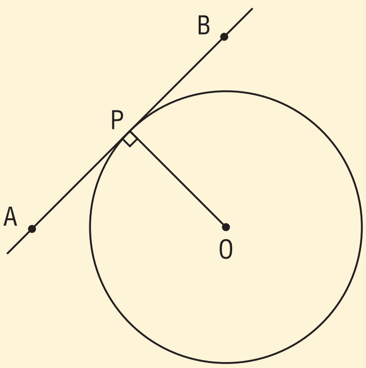
A line that intersects a circle at only one point is a **tangent** to the circle.
The point where the tangent intersects the circle is the **point of tangency**.

Line AB is a tangent to the circle with centre O.
Point P is the point of tangency.



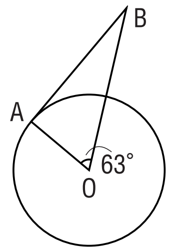
Tangent-Radius Property

A tangent to a circle is perpendicular to the radius at the point of tangency.
That is, ∠APO = ∠BPO = 90°



Example 1 Determining the Measure of an Angle in a Triangle

Point O is the centre of a circle
and AB is a tangent to the circle.
In ΔOAB, $\angle$AOB = 63°
Determine the measure of $\angle$OBA.



A Solution

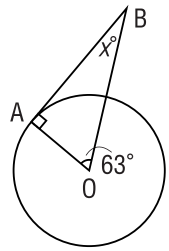
Let $x°$ represent the measure of $\angle$OBA.
Since AB is a tangent to the circle, $\angle$OAB = 90°
The sum of the angles in ΔOAB is 180°.

So, $x° + 90° + 63° = 180°$

$$x° = 180° - 90° - 63°$$

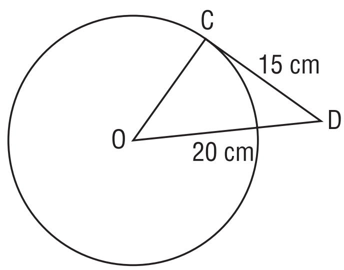
$$= 27°$$

So, $\angle$OBA = 27°



Example 2 Using the Pythagorean Theorem in a Circle

Point O is the centre of a circle
and CD is a tangent to the circle.
CD = 15 cm and OD = 20 cm
Determine the length of the radius OC.
Give the answer to the nearest tenth.



A Solution

Since CD is a tangent, $\angle$OCD = 90°
Use the Pythagorean Theorem in right ΔOCD to calculate OC.
Let d represent the length of OC.

$$d^2 + \text{CD}^2 = \text{OD}^2$$

$$d^2 + 15^2 = 20^2$$

$$d^2 + 225 = 400$$

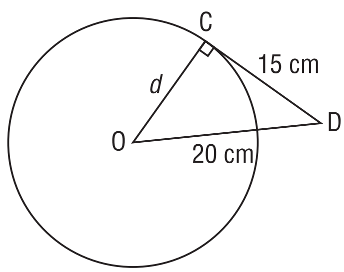
$$d^2 = 400 - 225$$

$$d^2 = 175$$

$$d = \sqrt{175}$$

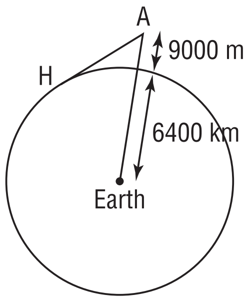
$$d \doteq 13.23$$

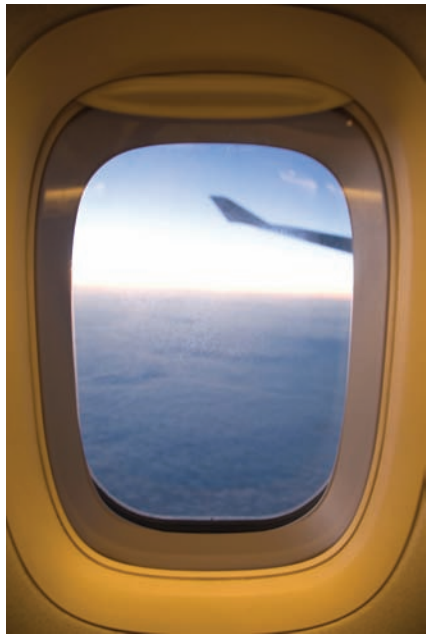
The radius of the circle is about 13.2 cm long.



Example 3 Solving Problems Using the Tangent and Radius Property

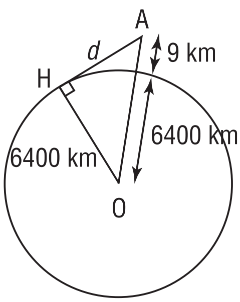
An airplane, A, is cruising at an altitude of 9000 m. A cross section of Earth is a circle with radius approximately 6400 km. A passenger wonders how far she is from a point H on the horizon she sees outside the window. Calculate this distance to the nearest kilometre.





A Solution

The line of sight AH from the passenger to the horizon is a tangent to the circle at H.



9000 m = 9 km

Since the tangent AH is perpendicular to the radius OH at the point of tangency H, ΔAHO is a right triangle, with $\angle$OHA = 90°.
Use the Pythagorean Theorem to calculate AH.
Let d represent the length of AH.

$$d^2 + 6400^2 = (6400 + 9)^2$$
$$d^2 + 6400^2 = 6409^2$$
$$d^2 = 6409^2 - 6400^2$$
$$d = \sqrt{6409^2 - 6400^2}$$
$$d \doteq 339.53$$

The passenger is about 340 km from the horizon.

Discuss the ideas

1. A line may look as if it is a tangent to a circle but it may not be. How can you determine if the line is a tangent?

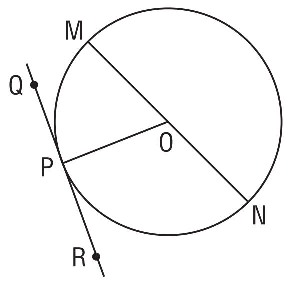
2. The Pythagorean Theorem was used in *Examples 2* and *3*. When is the Pythagorean Theorem useful for solving problems involving tangents?

Practice

Check

3. In each diagram, point O is the centre of each circle. Which lines are tangents?

a)



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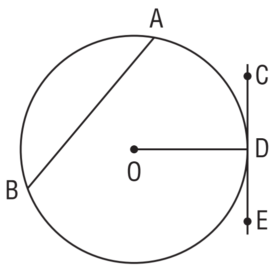
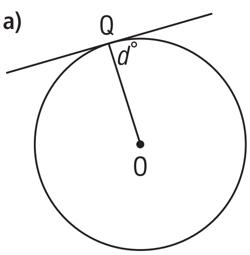


4. Point Q is a point of tangency. Point O is the centre of each circle. What is each value of $d°$?

a)



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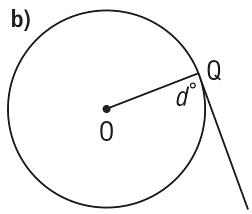
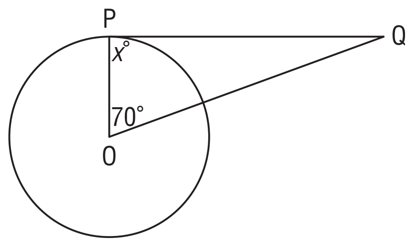


5. Point P is a point of tangency and O is the centre of each circle. Determine each value of $x°$.

a)



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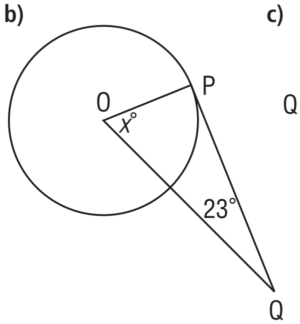
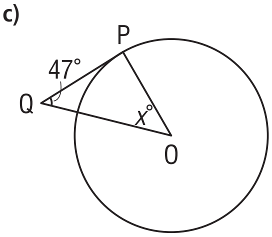
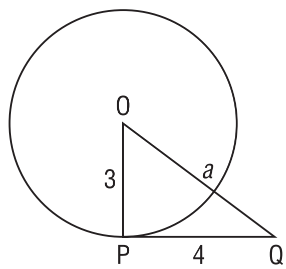


c)



6. Point P is a point of tangency and O is the centre of each circle. Determine each value of *a*.

a)



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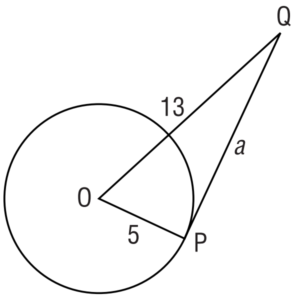

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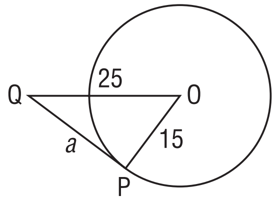
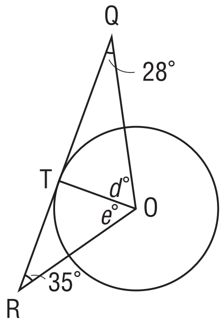


Apply

7. Point T is a point of tangency and O is the centre of each circle. Determine each value of *d*° and *e*°.

a)



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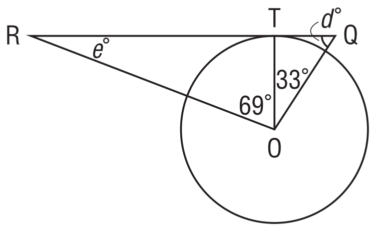
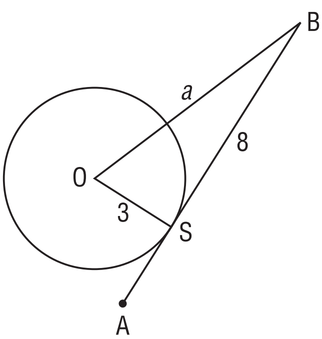


8. Point S is a point of tangency and O is the centre of each circle. Determine each value of *a* to the nearest tenth.

a)



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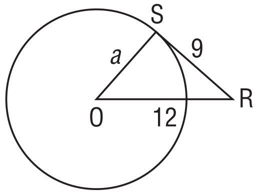
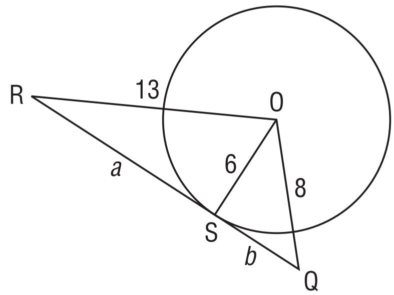
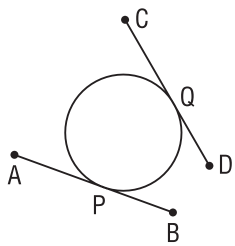


9. Point S is a point of tangency and O is the centre of the circle. Determine the values of *a* and *b* to the nearest tenth.

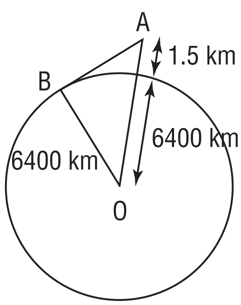


10. Look around the classroom or think of what you might see outside the classroom. Provide an example to illustrate that the tangent to a circle is perpendicular to the radius at the point of tangency.

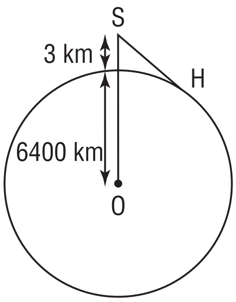
11. Both AB and CD are tangents to a circle at P and Q. Use what you know about tangents and radii to explain how to locate the centre of the circle. Justify your strategy.



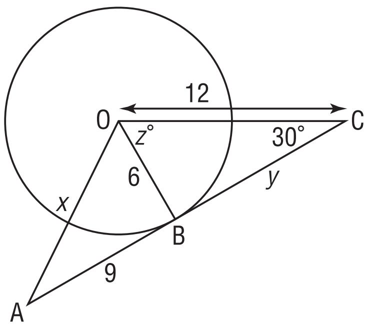
12. A small aircraft, A, is cruising at an altitude of 1.5 km. The radius of Earth is approximately 6400 km. How far is the plane from the horizon at B? Calculate this distance to the nearest kilometre.



13. A skydiver, S, jumps from a plane at an altitude of 3 km. The radius of Earth is approximately 6400 km. How far is the horizon, H, from the skydiver when she leaves the plane? Calculate this distance to the nearest kilometre.



14. Point O is the centre of the circle. Point B is a point of tangency. Determine the values of x, y, and $z°$. Give the answers to the nearest tenth where necessary. Justify the strategies you used.



15. Assessment Focus

a) From any point outside a circle, how many tangents do you think you can draw to the circle?
Explain your reasoning.

b) Construct a circle. Choose a point outside the circle. Check your answer to part a. How do you know you have drawn as many tangents as you can?

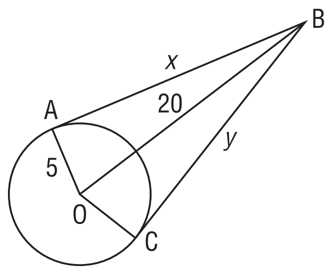
c) How do you know that the lines you have drawn are tangents?

Show your work.

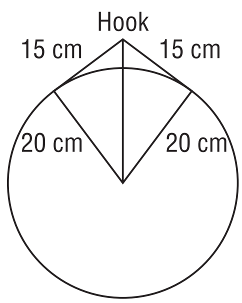
16. a) Construct a circle and draw two radii. Draw a tangent from the endpoint of each radius so the two tangents intersect at point N. Measure the distance from N to each point of tangency.
What do you notice?

b) Compare your answer to part a with that of your classmates. How do the lengths of two tangents drawn to a circle from the same point outside the circle appear to be related?

c) Points A and C are points of tangency and O is the centre of the circle. Calculate the values of x and y to the nearest tenth.
Do the answers confirm your conclusions in part b? Explain.



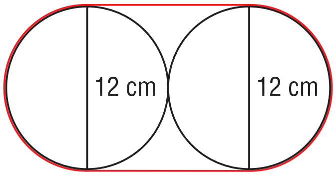
17. A circular mirror with radius 20 cm hangs by a wire from a hook. The wire is 30 cm long and is a tangent to the mirror in two places. How far above the top of the mirror is the hook? How do you know?



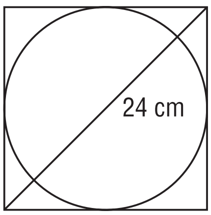
18. A communications satellite orbits Earth at an altitude of about 600 km. What distance from the satellite is the farthest point on Earth's surface that could receive its signal?
Justify the strategy you used.

Take It Further

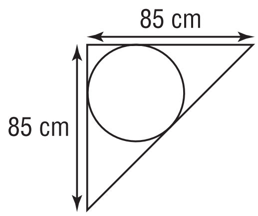
19. Two cylindrical rods are bound with a strap. Each rod has diameter 12 cm. How long is the strap? Give the answer to the nearest tenth of a centimetre. (The circumference C of a circle with diameter d is given by $C = \pi d$.)



20. What is the radius of the largest circle that can be cut from a square piece of paper whose diagonal is 24 cm long?



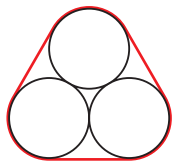
21. A cylindrical pipe touches a wall and the ceiling of a room. The pipe is supported by a brace. The ends of the brace are 85 cm from the wall and ceiling. Apply what you discovered in question 16. What is the diameter of the pipe? Give the answer to the nearest centimetre.



22. Each of 3 logs has diameter 1 m.

a) What is the minimum length of strap needed to wrap the logs?

b) Would this minimum length be the actual length of strap used? Explain.



Reflect

What do you know about a tangent and a radius in a circle? How can you use this property? Include examples in your explanation.

Math Link

Literacy

Sometimes a conversation goes off topic when the subject being discussed makes one person think of a related idea. For example, a discussion about Olympic athletes may prompt someone to think of and describe her exercise plan. When this happens, we say the discussion has "gone off on a tangent." How does this everyday occurrence relate to the meaning of the word "tangent" in math?

8.2 Properties of Chords in a Circle

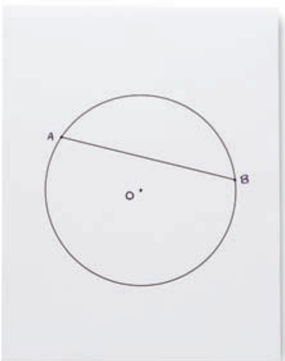
FOCUS

- Relate a chord, its perpendicular bisector, and the centre of the circle, then solve problems.

These pictures show the sun setting.
Imagine the sun as it touches the horizon.
How is the centre of the sun related to the horizon?

Investigate

2



You will need scissors, a compass, protractor, and ruler.

➤ Construct then cut out a large circle. Label the centre of the circle O.

➤ Choose two points A and B on the circle. Join these points to form line segment AB. Make sure AB does *not* go through the centre of the circle.

➤ Fold the circle so that A coincides with B. Crease the fold, open the circle, and draw a line along the fold. Mark the point C where the fold line intersects AB.
What do you notice about the angles at C?
What do you notice about line segments AC and CB?

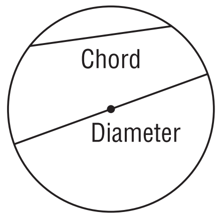
➤ Repeat the steps above for two other points D and E on the circle.

Reflect & Share

Compare your results with those of another pair of classmates.
What appears to be true about each line segment and its related fold line?
What name could you give each fold line?
Through which point do both fold lines appear to pass?

Connect

A line segment that joins two points on a circle is a **chord**.
A diameter of a circle is a chord through the centre of the circle.

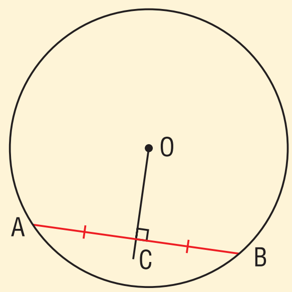


The chord, its perpendicular bisector, and the centre of the circle are related.

> A perpendicular bisector intersects a line segment at 90° and divides the line segment into two equal parts.

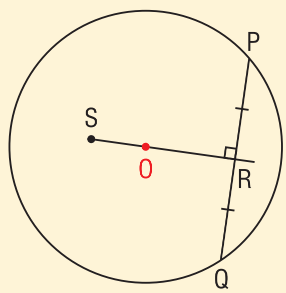
Perpendicular to Chord Property 1

The perpendicular from the centre of a circle to a chord bisects the chord; that is, the perpendicular divides the chord into two equal parts.
Point O is the centre of the circle.
When $\angle OCB = \angle OCA = 90°$,
then $AC = CB$



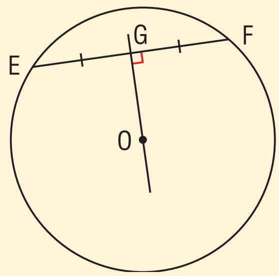
Perpendicular to Chord Property 2

The perpendicular bisector of a chord in a circle passes through the centre of the circle.
When $\angle SRP = \angle SRQ = 90°$ and $PR = RQ$,
then SR passes through O, the centre of the circle.



Perpendicular to Chord Property 3

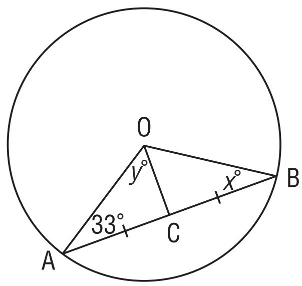
A line that joins the centre of a circle and the midpoint of a chord is perpendicular to the chord.
When O is the centre of a circle and EG = GF, then ∠OGE = ∠OGF = 90°



We can use these 3 properties to solve problems involving chords in a circle.

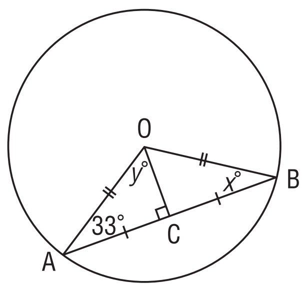
Example 1 Determining the Measure of Angles in a Triangle

Point O is the centre of a circle, and line segment OC bisects chord AB.
∠OAC = 33°
Determine the values of $x°$ and $y°$.



A Solution

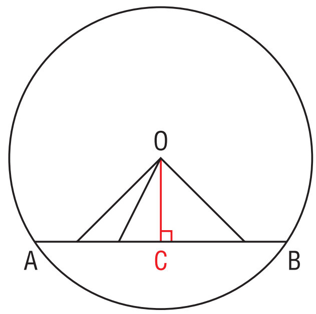
Since OC bisects the chord and passes through the centre of the circle, OC is perpendicular to AB.
So, ∠ACO = 90°
And, since the radii are equal, OA = OB, ΔOAB is isosceles.



Since ΔOAB is isosceles, then
∠OBA = ∠OAB
So, $x° = 33°$
In ΔOAC, use the sum of the angles in a triangle.

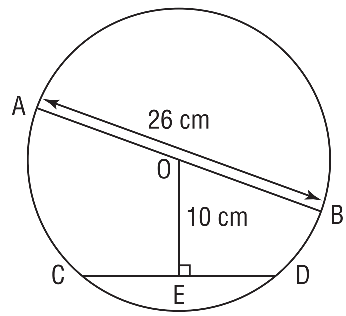
$y° + 33° + 90° = 180°$
$y° = 180° - 90° - 33°$
$= 57°$

Many line segments can be drawn from O, the centre of a circle, to a chord AB.
The *distance* from O to AB is defined as the shortest distance.
This distance is the length of the perpendicular from O to AB; that is, the length of OC.



Example 2 Using the Pythagorean Theorem in a Circle

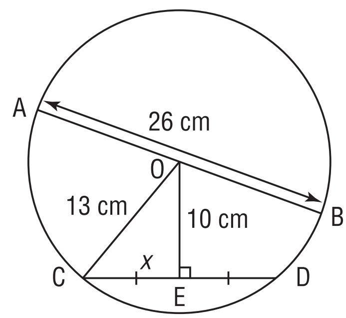
Point O is the centre of a circle.
AB is a diameter with length 26 cm.
CD is a chord that is 10 cm from the centre of the circle.
What is the length of chord CD?
Give the answer to the nearest tenth.



A Solution

The distance of a chord from the centre of a circle is the perpendicular distance from the centre to the chord.
Since OE is perpendicular to chord CD, then OE bisects CD, and CE = ED
To use the Pythagorean Theorem, join OC to form right ΔOCE.
OC is a radius, so OC is $\frac{1}{2}$ of AB, which is $\frac{1}{2}$ of 26 cm, or 13 cm.
Use the Pythagorean Theorem in ΔOCE to calculate CE.
Let the length of CE be represented by x.

$$\begin{aligned} OC^2 &= x^2 + OE^2 \\ 13^2 &= x^2 + 10^2 \\ 169 &= x^2 + 100 \\ 169 - 100 &= x^2 \\ 69 &= x^2 \\ x &= \sqrt{69} \\ x &\doteq 8.307 \end{aligned}$$



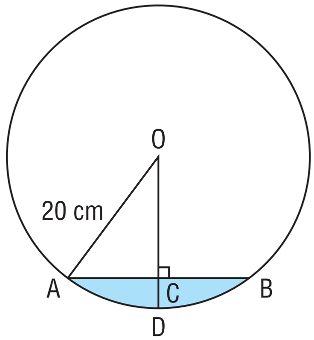
So, CE $\doteq$ 8.307 cm

$$\begin{aligned} \text{Chord CD} &= 2 \times \text{CE} \\ &\doteq 2 \times 8.307 \text{ cm} \\ &= 16.614 \text{ cm} \end{aligned}$$

Chord CD is about 16.6 cm long.

Example 3 Solving Problems Using the Property of a Chord and its Perpendicular

A horizontal pipe has a circular cross section, with centre O. Its radius is 20 cm.
Water fills less than one-half of the pipe.
The surface of the water AB is 24 cm wide.
Determine the maximum depth of the water, which is the depth CD.



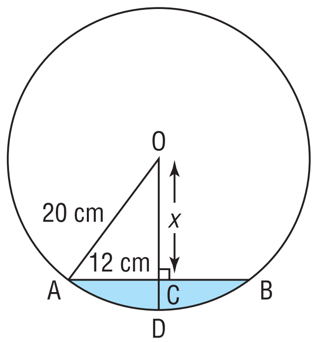
A Solution

The depth CD = OD − OC
OD is the radius, 20 cm.

Since OC is perpendicular to AB, then AC = $\frac{1}{2}$ of AB, which is $\frac{1}{2}$ of 24 cm, or 12 cm.

To determine OC, use the Pythagorean Theorem in ΔOAC.
Let x represent the length of OC.

$$\begin{aligned} AC^2 + x^2 &= OA^2 \\ 12^2 + x^2 &= 20^2 \\ 144 + x^2 &= 400 \\ x^2 &= 400 - 144 \\ x^2 &= 256 \\ x &= \sqrt{256} \\ x &= 16 \end{aligned}$$



$$\begin{aligned} CD &= OD - OC \\ &= 20\text{ cm} - 16\text{ cm} \\ &= 4\text{ cm} \end{aligned}$$

The maximum depth of the water is 4 cm.

Discuss the ideas

1. In a circle, how are these 3 items related?
- the centre of the circle
- a chord of a circle
- the perpendicular bisector of the chord

2. A diameter of a circle is a chord of the circle. How is the answer to question 1 affected if the chord is a diameter?

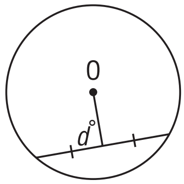
Practice

Check

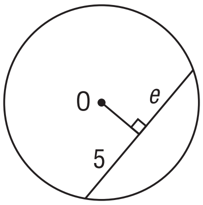
Give the answers to the nearest tenth where necessary.

3. Point O is the centre of each circle. Determine the values of $d°$, e, and f.

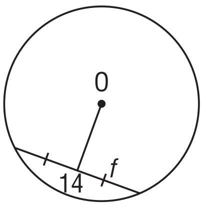
a)



b)

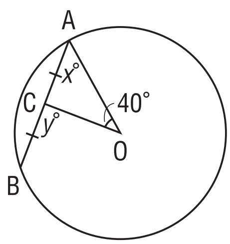


c)



4. Point O is the centre of each circle. Determine each value of $x°$ and $y°$.

a)



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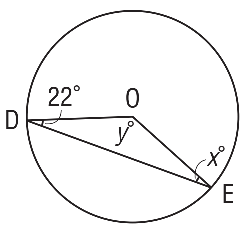
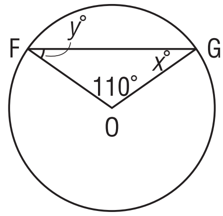
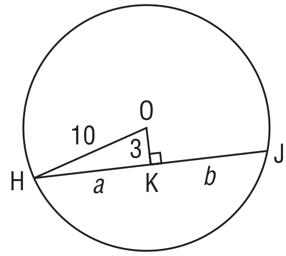


c)



5. Point O is the centre of each circle. Determine each value of a and b.

a)



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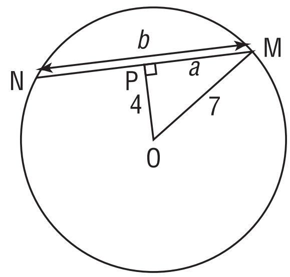
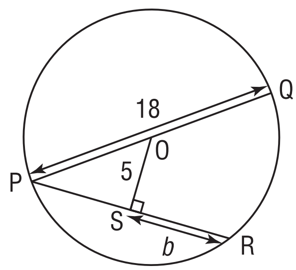


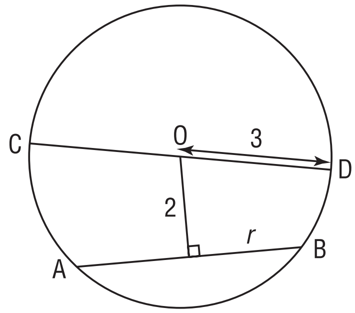
Apply

6. Point O is the centre of the circle. Determine the value of b. Which circle properties did you use?

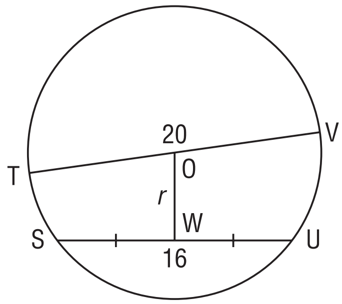


7. Point O is the centre of each circle. Determine each value of *r*. Which extra line segments do you need to draw first? Justify your solutions.

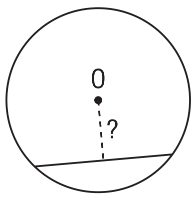
a)



b)



8. Construct a large circle, centre O.



a) Draw, then measure a chord in the circle. How far is the chord from O?

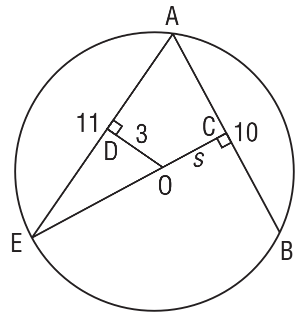
b) Draw other chords that are the same length as the chord you drew in part a. For each chord you draw, measure its distance from O. What do you notice?

c) Compare your results with those of other students. What appears to be true about congruent chords in a circle?

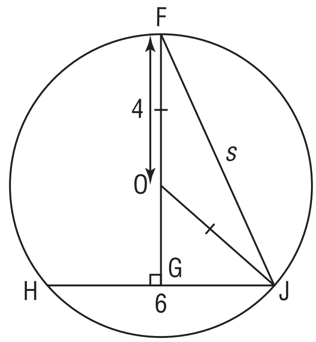
9. Trace a circular object to draw a circle without marking its centre. Draw two chords in the circle. Use what you have learned in this lesson to locate the centre of the circle. Justify your strategy.

10. Point O is the centre of each circle. Determine each value of *s*. Which circle properties did you use?

a)



b)



11. A circle has diameter 25 cm. How far from the centre of this circle is a chord 16 cm long? Justify your answer.

12. Assessment Focus

A circle has diameter 14 cm.

a) Which of the following measures could be lengths of chords in this circle? Justify your answers. How could you check your answers?

i) 5 cm **ii)** 9 cm

iii) 14 cm **iv)** 18 cm

b) For each possible length you identified in part a, determine how far the chord is from the centre of the circle.

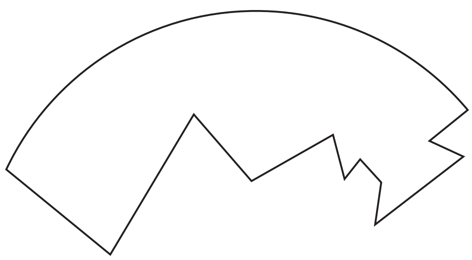
Show your work. State which circle properties you used.

13. Draw and label a diagram to illustrate that the perpendicular to a chord from the centre of a circle bisects the chord.

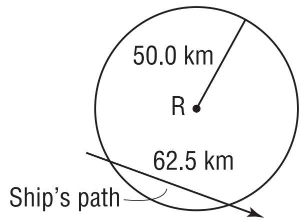
14. A chord is 6 cm long. It is 15 cm from the centre of a circle. What is the radius of the circle?

15. A circle has diameter 13 cm. In the circle, each of two chords is 8 cm long.
   a) What is the shortest distance from each chord to the centre of the circle?
   b) What do you notice about these congruent chords?

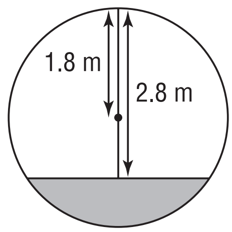
16. An archaeologist discovers a fragment of a circular plate on a dig at a prehistoric site. She wants to sketch the missing portion of the plate to determine how large it was. Trace the image of the plate fragment. Locate the centre of the plate. Use a compass to complete the sketch of the plate. Explain your work.



17. A radar station R tracks all ships in a circle with radius 50.0 km. A ship enters this radar zone and the station tracks it for 62.5 km until the ship passes out of range. What is the closest distance the ship comes to the radar station? Justify your answer.



18. A pedestrian underpass is constructed beneath a roadway using a cylindrical pipe with radius 1.8 m. The bottom of the pipe will be filled and paved. The headroom at the centre of the path is 2.8 m.
How wide is the path?



Take It Further

19. A spherical fish bowl has diameter 26 cm. The surface of the water in the bowl is a circle with diameter 20 cm.
   a) What is the maximum depth of the water?
   b) How many different answers are there for part a? Explain.

Reflect

What is the relationship among the centre of a circle, a chord, and the perpendicular bisector of the chord? Use an example to show how this relationship can help you calculate some measures in circles.

Verifying the Tangent and Chord Properties

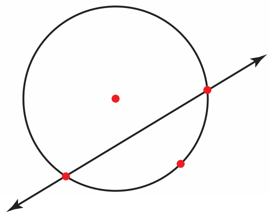
FOCUS

- Use dynamic geometry software to verify the tangent and chord properties in a circle.

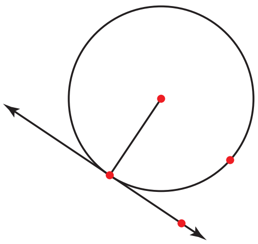
Dynamic geometry software on a computer or a graphing calculator can be used to verify the circle properties in Lessons 8.1 and 8.2.
The diagrams show what you might see as you conduct the investigations that follow.

To verify the tangent property

1. Construct a circle. If the software uses a point on the circle to define the circle, make sure you do not use this point for any further steps.
2. Construct a point on the circle.
   If the software has a "Draw Tangent" tool, use it to draw a tangent at the point you constructed.

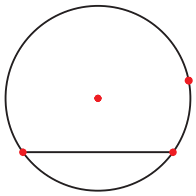
   Otherwise, construct another point on the circle. Construct a line that intersects the circle at the two points.
   Drag one of the two points on the line until it coincides with the other point. The line is now a tangent to the circle.
3. Construct a line segment to join the centre of the circle and the point of tangency. What does this line segment represent?



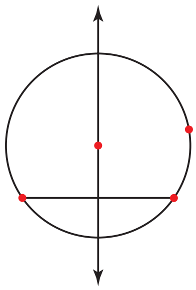
4. To measure the angle between the tangent and radius, you need a second point on the tangent. Construct a second point if the software does not do this automatically.
5. Use the software's measurement tools to measure the angle between the radius and tangent. Does the angle measure match what you have learned about a tangent and radius? If not, suggest a reason why.
6. Draw other tangents to the circle and the radii that pass through the points of tangency. Measure the angle between each tangent and radius. What do you notice?
7. Drag either the circle or its centre to investigate the property for circles of different sizes. What is always true about the angle between a tangent to a circle and a radius at the point of tangency?

To verify the chord property

1. Construct a circle.
2. Construct a line segment to join two points on the circle.



3. Construct a line perpendicular to the segment through the centre of the circle.



4. Use measurement tools to measure the distance between each endpoint of the chord and the point of intersection of the chord and the perpendicular. What do you notice?
5. Drag the endpoints of the chord to different positions on the circle to check the results for other chords. What do you notice?
6. Drag the circle or its centre to investigate the property for circles of different sizes. What is always true about a perpendicular from the centre of a circle to a chord in the circle?

Check

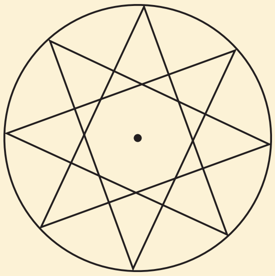
1. Suppose you have constructed a circle and one of its radii. Suggest a way, different from those mentioned above, to construct a tangent at the endpoint of the radius. Construct several circles, radii, and tangents to demonstrate your ideas.
2. Use the geometry software to verify these properties of chords.
   a) Construct a chord. Construct its midpoint. Construct a perpendicular through the midpoint. Through which point does the perpendicular bisector of the chord pass?
   b) Construct a chord. Construct its midpoint.
   Construct a line segment from the midpoint to the centre of the circle. What is the measure of the angle between the chord and this line segment?

GAME

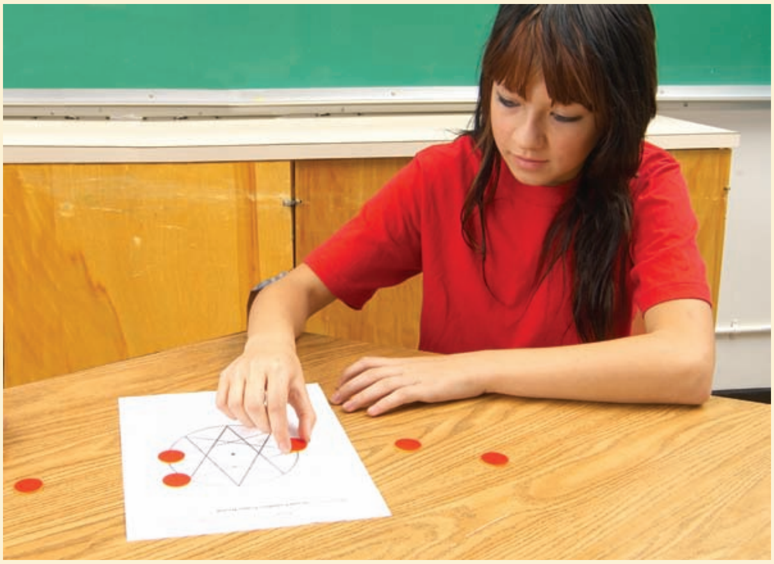
Seven Counters

How to Play

Your teacher will give you a larger copy of this game board:



1. Place a counter on a vertex of the star in the game board. Slide the counter along any segment of the star to place it at another vertex. The counter is now fixed.
2. Continue placing counters by sliding them from one vertex to another until no more counters can be placed. A counter cannot be placed on the game board if no vertex has a line segment along which the counter can move.
3. If you play with a partner, take turns to place counters and work together to decide on a winning strategy.
4. If you play against a partner, work independently to see who can place more counters.
5. It is possible to place all seven counters on the board, leaving one vertex of the star open.
   Keep trying until you have succeeded!
6. Explain your winning strategy.



You will need

- a copy of the game board
- 7 counters

Number of Players

- 1 or 2

Goal of the Game

- To place all seven counters on the game board

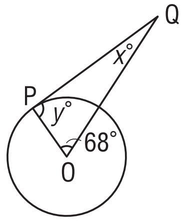
Mid-Unit Review

Give the answers to the nearest tenth where necessary.

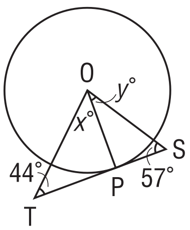
8.1

1. Point O is the centre of each circle and P is a point of tangency. Determine each value of $x°$ and $y°$.
Which circle properties did you use?

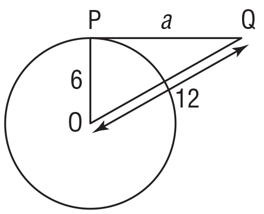
a)



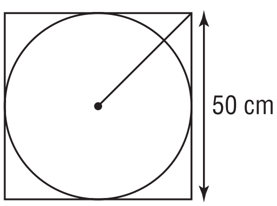
b)



2. Point O is the centre of a circle and point P is a point of tangency. Determine the value of a. Explain your strategy.

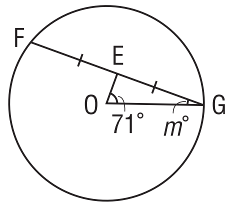


3. A metal disc is to be cut from a square sheet with side length 50 cm.
How far from a corner of the sheet is the centre of the disc? Justify your strategy.



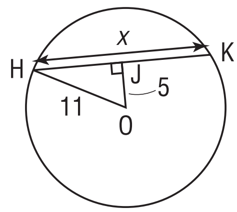
8.2

4. Point O is the centre of the circle. Determine the value of $m°$.



5. Point O is the centre of each circle. Determine each value of x.

a)



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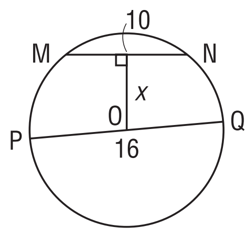
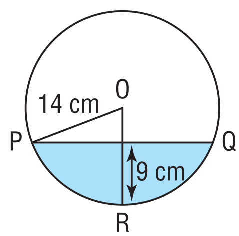


6. A circle has diameter 32 cm. A chord AB is 6 cm from O, the centre of the circle.

a) Sketch a diagram.

b) What is the length of the chord? Which circle properties did you use to find out?

7. Water is flowing through a pipe with radius 14 cm. The maximum depth of the water is 9 cm. What is the width, PQ, of the surface of the water?



8.3 Properties of Angles in a Circle

FOCUS

- Discover the properties of inscribed angles and central angles, then solve related problems.

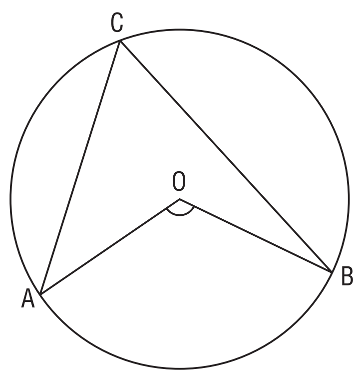
A soccer player attempts to get a goal. In a warm-up, players line up parallel to the goal line to shoot on the net. Does each player have the same shooting angle? Is there an arrangement that allows the players to be spread out but still have the same shooting angle?

Investigate

2

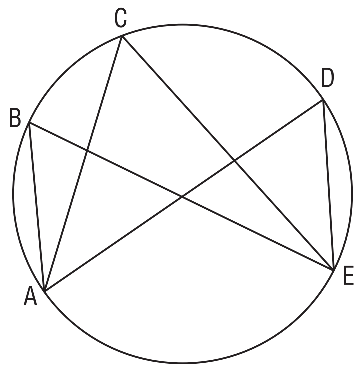
You will need a compass, ruler, and protractor.

➤ Construct a large circle, centre O.
Choose two points A and B on the circle.
Choose a third point C on the circle.
Join AC and BC.
Measure ∠ACB. Join AO and OB.
Measure the smaller ∠AOB.
Record your measurements.



➤ Repeat the previous step for other points A, B, and C on the circle and for other circles.

➤ Construct another large circle.
Mark 5 points A, B, C, D, and E, in order, on the circle.
Join AB, AC, AD, and EB, EC, ED.
Measure ∠ABE, ∠ACE, and ∠ADE.
Record your measurements.



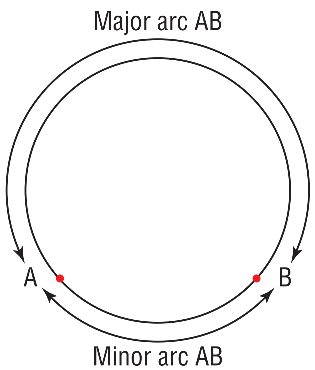
➤ Repeat the previous step for other circles.

Reflect & Share

Compare your results with those of another pair of students.
What relationship did you discover between the angle at the centre of a circle and the angle on the circle?
What relationship did you discover among the angles on a circle?

Connect

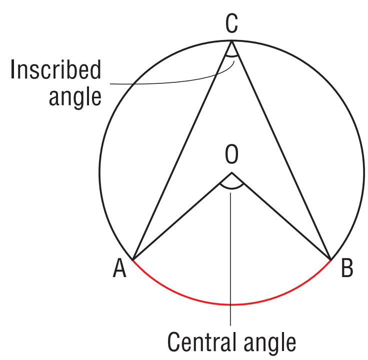
➤ A section of the circumference of a circle is an **arc**.
The shorter arc AB is the **minor arc**.
The longer arc AB is the **major arc**.



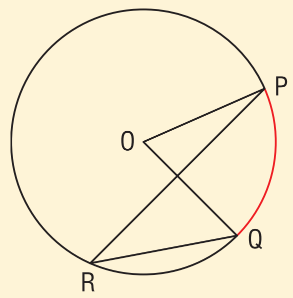
➤ The angle formed by joining the endpoints of an arc to the centre of the circle is a **central angle**;
∠AOB is a central angle.

The angle formed by joining the endpoints of an arc to a point on the circle is an **inscribed angle**;
∠ACB is an inscribed angle.

The inscribed and central angles in this circle are **subtended** by the minor arc AB.

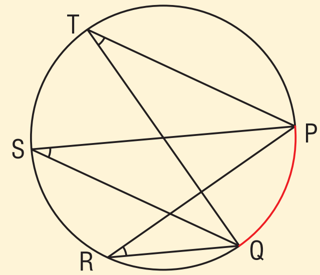


> **Central Angle and Inscribed Angle Property**
> In a circle, the measure of a central angle subtended by an arc is twice the measure of an inscribed angle subtended by the same arc.
> $\angle POQ = 2\,\angle PRQ$, or
> $\angle PRQ = \frac{1}{2}\,\angle POQ$

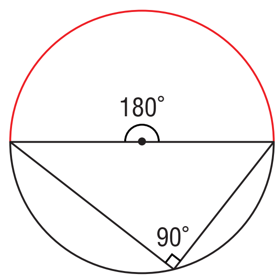


The above property is true for any inscribed angle.

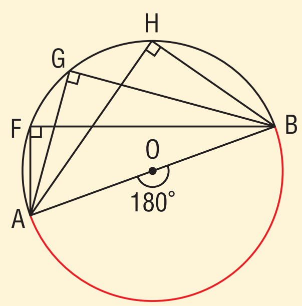
> **Inscribed Angles Property**
> In a circle, all inscribed angles subtended by the same arc are congruent.
> $\angle PTQ = \angle PSQ = \angle PRQ$



➤ The two arcs formed by the endpoints of a diameter are semicircles.
The central angle of each arc is a straight angle, which is 180°.
The inscribed angle subtended by a semicircle is one-half of 180°, or 90°.



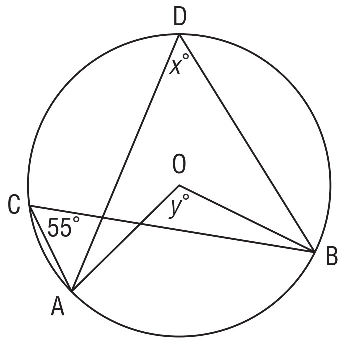
> **Angles in a Semicircle Property**
> All inscribed angles subtended by a semicircle are right angles.
> Since $\angle AOB = 180°$,
> then $\angle AFB = \angle AGB = \angle AHB = 90°$



We say: The angle *inscribed* in a semicircle is a right angle.
We also know that if an inscribed angle is 90°, then it is subtended by a semicircle.

Example 1 Using Inscribed and Central Angles

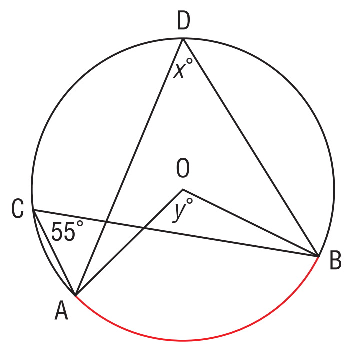
Point O is the centre of a circle.
Determine the values of $x°$ and $y°$.



A Solution

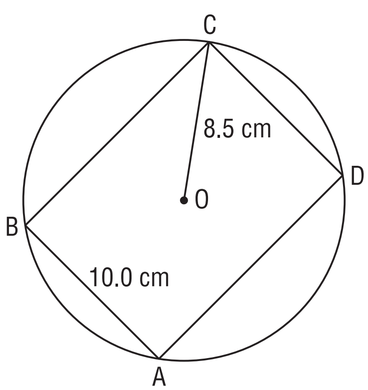
Since ∠ADB and ∠ACB are inscribed angles subtended by the same arc AB, these angles are congruent.
So, $x° = 55°$
Both the central ∠AOB and the inscribed ∠ACB are subtended by minor arc AB.
So, the central angle is twice the inscribed angle.
That is, ∠AOB = 2∠ACB

$$y° = 2 \times 55°$$
$$= 110°$$



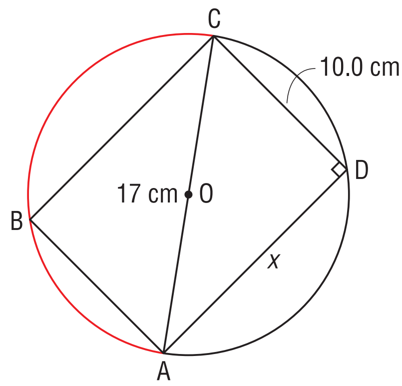
Example 2 Applying the Property of an Angle Inscribed in a Semicircle

Rectangle ABCD has its vertices on a circle with radius 8.5 cm.
The width of the rectangle is 10.0 cm. What is its length?
Give the answer to the nearest tenth.



A Solution

The length of the rectangle is AD.
Each angle of the rectangle is 90°.
So, each angle is subtended by a semicircle:
∠ADC is subtended by semicircle ABC.
This means that each diagonal of the rectangle is a diameter of the circle:
AC is a diameter.



The radius of the circle is 8.5 cm,
so AC = 2 × 8.5 cm, or 17 cm.

Use the Pythagorean Theorem in right ΔADC to calculate AD.
Let the length of AD be represented by x.

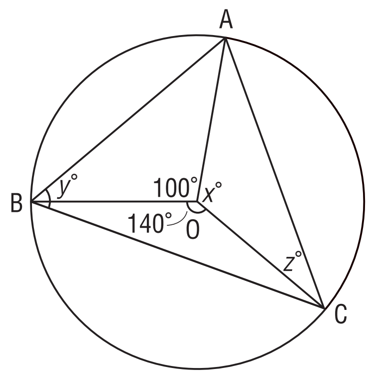
$$x^2 + CD^2 = AC^2$$
$$x^2 + 10^2 = 17^2$$
$$x^2 + 100 = 289$$
$$x^2 = 289 - 100$$
$$x^2 = 189$$
$$x = \sqrt{189}$$
$$\doteq 13.748$$

The rectangle is about 13.7 cm long.

A polygon whose vertices lie on a circle is an **inscribed polygon**.
In *Example 2*, rectangle ABCD is an *inscribed rectangle*.
Rectangle ABCD is *inscribed* in the circle.

Example 3 Determining Angles in an Inscribed Triangle

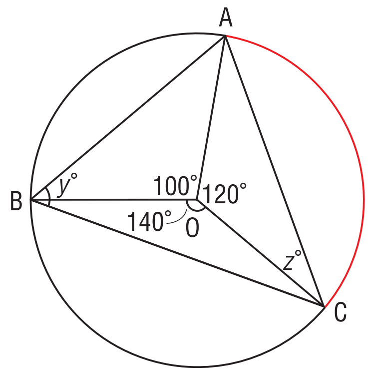
Triangle ABC is inscribed in a circle, centre O.
∠AOB = 100° and ∠COB = 140°
Determine the values of $x°$, $y°$, and $z°$.



A Solution

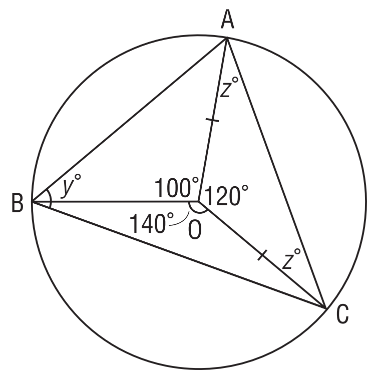
The sum of the central angles in a circle is 360°.

$$
\begin{aligned}
\text{So, } 100° + 140° + x° &= 360° \\
240° + x° &= 360° \\
x° &= 360° - 240° \\
&= 120°
\end{aligned}
$$



∠ABC is an inscribed angle and ∠AOC is a central angle subtended by the same arc.

$$
\begin{aligned}
\text{So, } \angle ABC &= \frac{1}{2}\angle AOC \\
y° &= \frac{1}{2} \times 120° \\
&= 60°
\end{aligned}
$$



OA and OC are radii, so ΔOAC is isosceles, with ∠OAC = ∠OCA = z°

The sum of the angles in a triangle is 180°.

So, in ΔOAC,

$$
\begin{aligned}
120° + z° + z° &= 180° \\
120° + 2z° &= 180° \\
2z° &= 180° - 120° \\
&= 60° \\
z° &= \frac{60°}{2} \\
&= 30°
\end{aligned}
$$

Discuss the ideas

1. How can the circle properties in this lesson help you decide where soccer players need to stand to have the same shooting angle on goal?

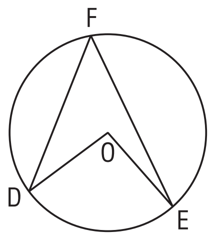
2. Suppose a circle has an inscribed angle. How do you identify the arc that subtends the angle?

Practice

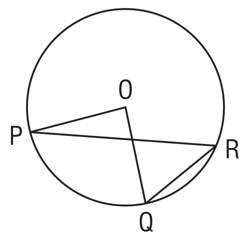
Check

3. In each circle, identify an inscribed angle and the central angle subtended by the same arc.

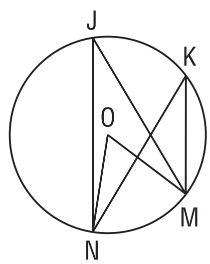
a)



b)

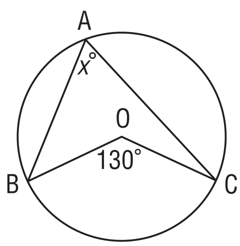


c)

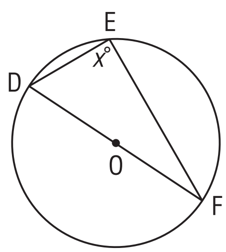


4. Point O is the centre of each circle. Determine each value of $x°$.

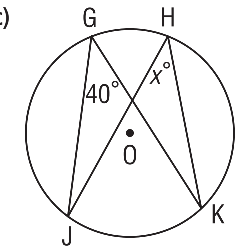
a)



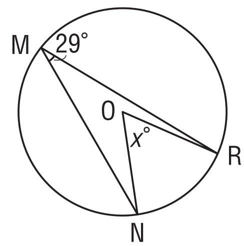
b)



c)



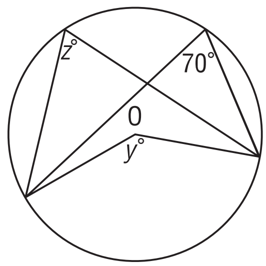
d)



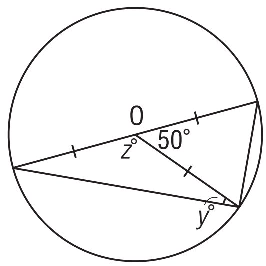
Apply

5. Point O is the centre of each circle. Label each vertex. Determine each value of $y°$ and $z°$. Which circle properties did you use?

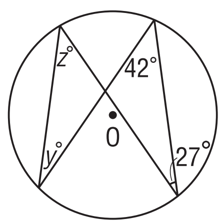
a)



b)

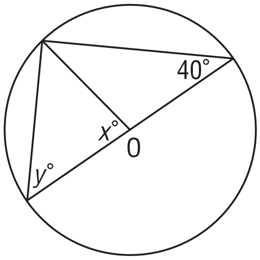


c)

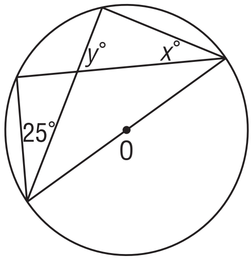


6. Point O is the centre of each circle. Label each vertex. Determine each value of $x°$ and $y°$. Which circle properties did you use?

a)



b)



7. Construct a circle and two diameters PR and QS. Join the endpoints of the diameters to form quadrilateral PQRS.

a) What type of quadrilateral is PQRS? Use what you have learned in this lesson to justify your answer.

b) What type of quadrilateral is PQRS when the diagonals are perpendicular? Construct a diagram to check your answer.

8. Draw and label a diagram to illustrate:

a) the measure of the central angle in a circle is equal to twice the measure of an inscribed angle subtended by the same arc

b) the inscribed angles subtended by the same arc of a circle are equal

9. Rectangle PQRS is inscribed in a circle with radius 7 cm. The length of the rectangle is 12 cm.

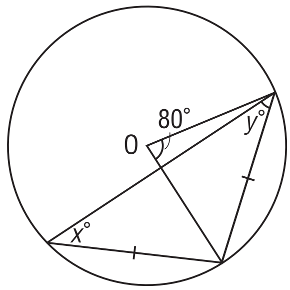
a) Sketch a diagram.

b) What is the width of the rectangle? Give the answer to the nearest tenth. Justify your solution.

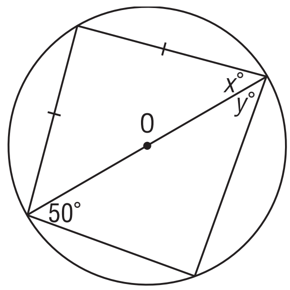
10. **Assessment Focus** Geometry sets often include *set squares*. A set square is a plastic right triangle. Trace around a circular object. Explain how you can use a set square and what you know about the angle in a semicircle to locate the centre of the circle. Justify your solution.

11. Point O is the centre of each circle. Label each vertex. Determine each value of $x°$ and $y°$. Which circle properties does each question illustrate?

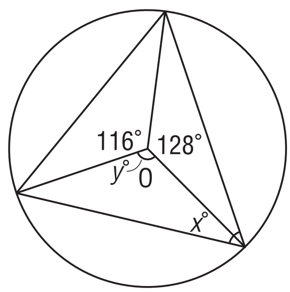
a)



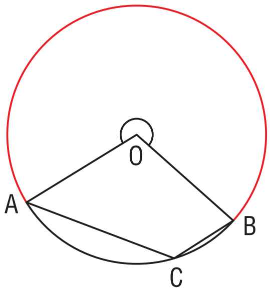
b)



c)



12. In *Investigate* on page 404, point C was on the major arc AB of a circle, centre O. Suppose C was on the minor arc AB. Do the circle properties that relate inscribed angles and central angles still apply? Investigate to find out. Justify your answer.



13. Some hockey players are approaching the goal. Two of them are the same distance from the end boards. Rana's shooting angle is 30° while Raji's is 35°.

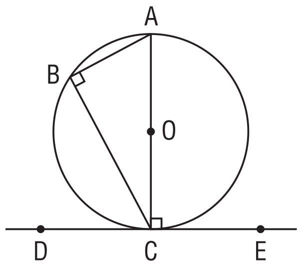
a) Sketch a diagram.

b) Who is closer to the middle of the ice? Explain your reasoning.

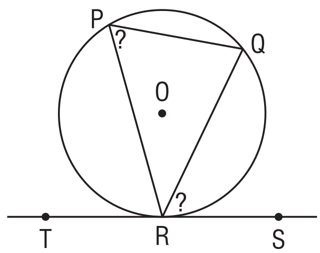
Take It Further

14. The *Seven Counters* game board on page 402 is an 8-pointed star inscribed in a circle. The vertices are equally spaced around the circle. What is the measure of the inscribed angle at each vertex of the star? Justify your solution.

15. The measure of ∠ACE between a tangent DE and the diameter AC at the point of tangency C is 90°. The measure of ∠ABC inscribed in a semicircle is also 90°.



a) How does the angle between a tangent and a chord appear to be related to the inscribed angle on the opposite side of the chord? That is, how is ∠QRS related to ∠QPR? Are ∠PRT and ∠PQR related in a similar way? Explain your reasoning.



b) Construct and measure accurate diagrams to verify the relationship in part a.

Reflect

Make a poster that summarizes the properties of angles in a circle.

Verifying the Angle Properties

FOCUS

- Use dynamic geometry software to verify the properties of angles in a circle.

Dynamic geometry software on a computer or a graphing calculator can be used to verify the circle properties in Lesson 8.3.

The diagrams show what you might see as you conduct the investigations that follow.

To verify the property of inscribed and central angles

1. Construct a circle.

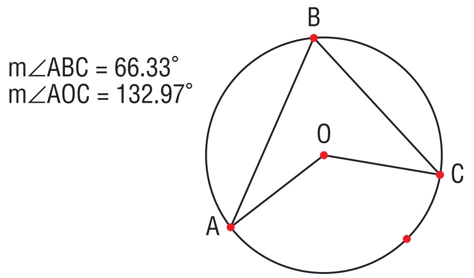
2. Mark three points on the circle.
Label them A, B, and C.
Label the centre of the circle O.

3. Join AB and BC. Join OA and OC.

4. Measure ∠ABC and ∠AOC.
What do you notice?

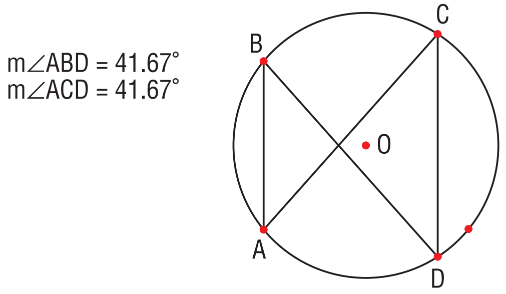
5. Drag point C around the circle. Do *not* drag it between points A and B.
Does the measure of ∠ABC change?
What property does this verify?

6. Drag point A or B around the circle. What do you notice about the angle measure relationship?



To verify the property of inscribed angles subtended by the same arc

1. Construct a circle.
2. Mark four points on the circle. Label them A, B, C, D in order. Label the centre of the circle O.
3. Join AB, AC, BD, and CD.
4. Measure ∠ABD and ∠ACD. What do you notice?
5. Drag point C around the circle. What do you notice about the angle measures? What property does this verify?



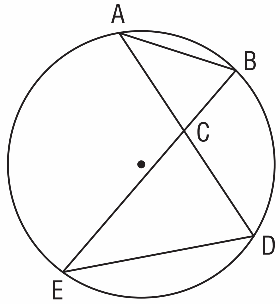
Check

1. In the first investigation, you dragged point C around the major arc AB. Predict what would happen if you dragged C to the minor arc AB. Use the software to confirm your prediction.
2. Use the software to confirm that all right triangles can be inscribed in a circle. Justify your strategy.

Start Where You Are

How Do I Best Learn Math?

Suppose I have to investigate two triangles like these in a circle.



➤ I could work alone or with others.

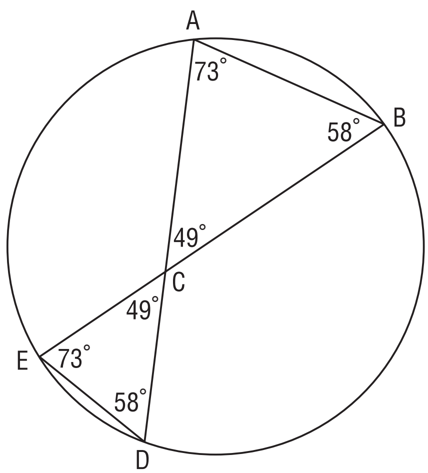
- Keena says, "I prefer to think things through on my own."
- Jetta says, "I like to discuss my ideas with a partner."
- Tyrell says, "I like to work in a group to get lots of ideas."

➤ I use what I know about angles.

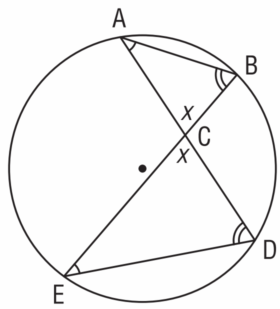
- Keena's method:
  I drew and labelled a diagram like the one above, then measured the angles in the triangles.
  I made sure that the diagram is big enough to be able to measure the angles with a protractor.
  I recorded the angle measures on the diagram.
  I noticed that pairs of angles in the two triangles are equal:

  $\angle ABE = \angle ADE = 58°$
  $\angle BAD = \angle BED = 73°$
  $\angle ACB = \angle ECD = 49°$

  So, the triangles are similar.
  $\Delta ABC \sim \Delta EDC$



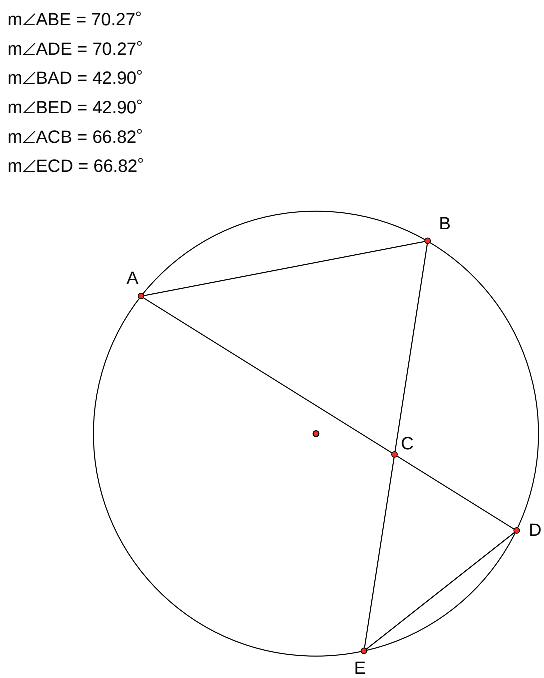
- Jetta's method:
  I reasoned from what I have learned
  about angles in a triangle.
  Arc AE subtends inscribed angles
  at B and at D.
  So, ∠ABE = ∠ADE
  Arc BD subtends inscribed angles
  at A and at E.
  So, ∠BAD = ∠BED
  Since two pairs of angles in the two triangles
  are equal, the angles in the third pair must also
  be equal, because the sum of the angles in any triangle is 180°.
  So, ∠ACB = ∠ECD
  Since 3 pairs of corresponding angles in two triangles are equal,
  the triangles are similar.
  ΔABC ~ ΔEDC



- Tyrell's method:
  I used geometry software.
  I drew a circle and two intersecting chords.
  I then joined the ends of the chords
  to form two triangles.
  I labelled the vertices of the triangles.
  I used the software to measure the angles.
  I rounded the angle measures shown
  on the screen to the nearest degree.

  From the screen, I noticed that
  these angles are equal:
  ∠ABE = ∠ADE = 70°
  ∠BAD = ∠BED = 43°
  ∠ACB = ∠ECD = 67°

  Since 3 pairs of corresponding angles in
  two triangles are equal,
  the triangles are similar.
  ΔABC ~ ΔEDC

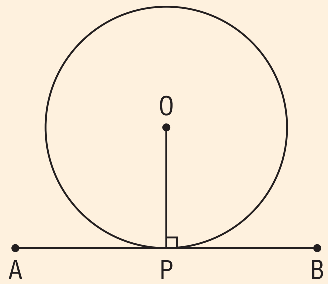


Check

1. Choose the way you best learn math.
   Investigate whether all rectangles can be inscribed in a circle.

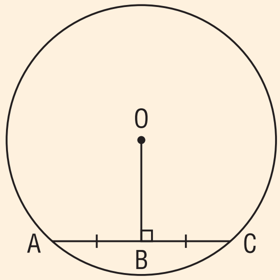
Study Guide

- A tangent to a circle is perpendicular to the radius at the point of tangency.
  That is, $\angle APO = \angle BPO = 90°$

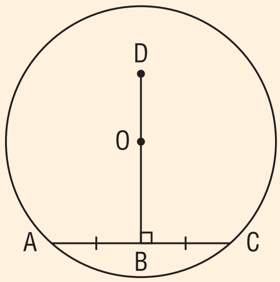


- The perpendicular from the centre of a circle to a chord bisects the chord.
  When $\angle OBC = \angle OBA = 90°$, then $AB = BC$

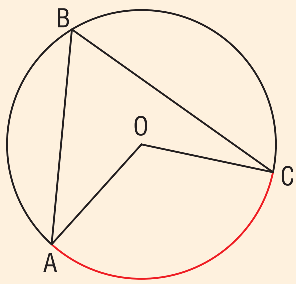
- A line segment that joins the centre of a circle to the midpoint of a chord is perpendicular to the chord.
  When O is the centre of a circle and $AB = BC$, then $\angle OBC = \angle OBA = 90°$



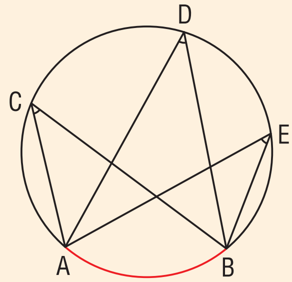
- The perpendicular bisector of a chord in a circle passes through the centre of the circle.
  When $\angle OBC = \angle OBA = 90°$, and $AB = BC$, then the centre O of the circle lies on DB.



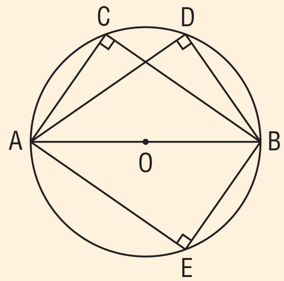
- The measure of a central angle subtended by an arc is twice the measure of an inscribed angle subtended by the same arc.
  $\angle AOC = 2\angle ABC$, or
  $\angle ABC = \frac{1}{2}\angle AOC$



- All inscribed angles subtended by same arc are congruent.
  $\angle ACB = \angle ADB = \angle AEB$



- All inscribed angles subtended by a semicircle are right angles.
$\angle ACB = \angle ADB = \angle AEB = 90°$



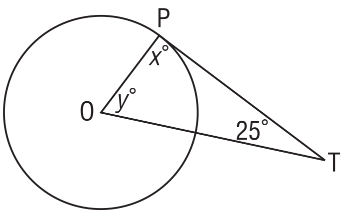
Review

Give the answers to the nearest tenth where necessary.

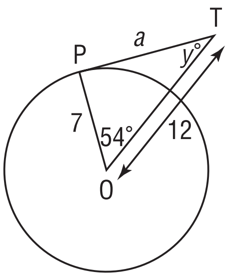
8.1

1. Point O is the centre of each circle. Segments PT and QT are tangents. Determine each value of $x°$, $y°$, a, and b. Show your work.

a)



b)



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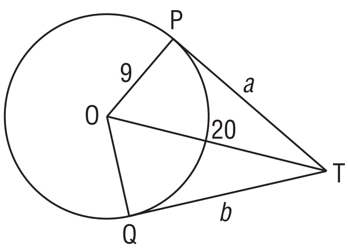
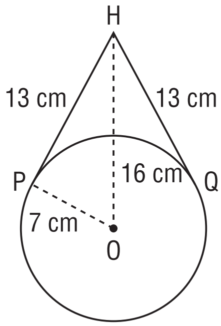
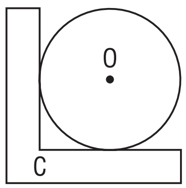


2. A circular mirror is suspended by a wire from a hook, H. Point O is the centre of the circle and is 16 cm below H. Explain how you know that the wire is *not* a tangent to the circle at P and at Q.



3. Draw a circle with centre O. Mark a point P on the circle. Explain how to draw a tangent to the circle. Which circle property did you use?

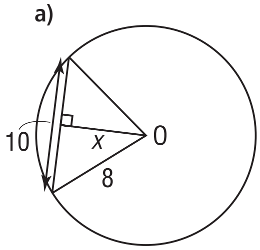
4. A circular plate is supported so it touches two sides of a shelf. The diameter of the plate is 20 cm. How far is the centre O of the plate from the inside corner C of the shelf? Which circle properties helped you find out?



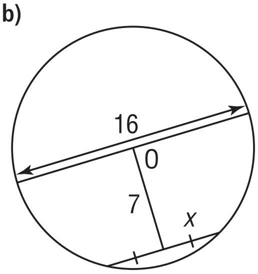
8.2

5. Point O is the centre of each circle.
Determine each value of *x*.
Justify your answers.

a)



b)



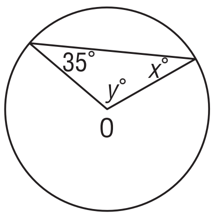
6. A dream catcher with diameter 22 cm is strung with a web of straight chords. One of these chords is 18 cm long.

a) Sketch a diagram.

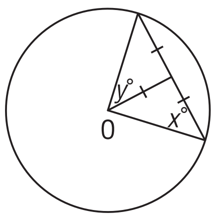
b) How far is the chord from the centre of the circle? Justify your solution strategy.

7. Point O is the centre of each circle.
Determine each value of $x°$ and $y°$.
Which circle properties did you use?

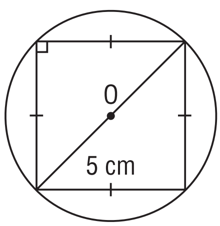
a)



b)



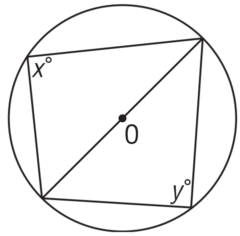
8. A square has side length 5 cm. It is inscribed in a circle, centre O.
What is the length of the radius of the circle? How do you know?



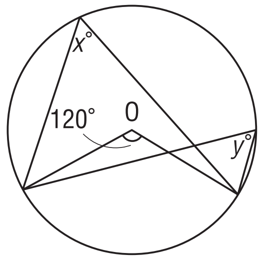
8.3

9. Point O is the centre of each circle.
Determine each value of $x°$ and $y°$.
Justify your answers.

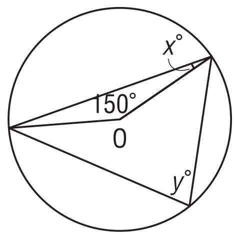
a)



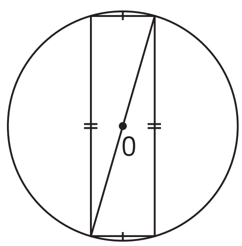
b)



c)

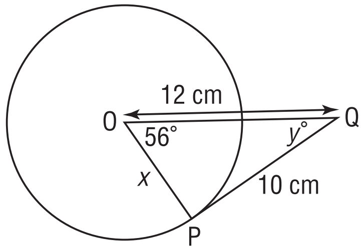


10. A rectangle is inscribed in a circle, centre O and diameter 36.0 cm. A shorter side of the rectangle is 10.0 cm long. What is the length of a longer side? How do you know?

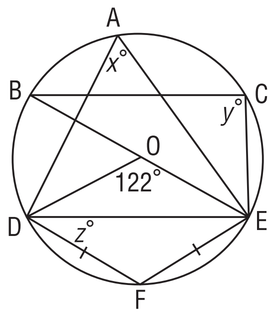


Practice Test

1. Point O is the centre of the circle. Point P is a point of tangency. Determine the values of x and $y°$. Give reasons for your answers.



2. Point O is the centre of the circle. Determine the values of $x°$, $y°$, and $z°$. Which circle properties did you use each time?



3. A circle has diameter 6.0 cm. Chord AB is 2.0 cm from the centre of the circle.

a) Sketch a diagram.

b) How long is the chord AB?

c) Another chord, CD, in the circle is 2.5 cm from the centre of the circle. Is chord CD longer or shorter than chord AB? Justify your answer.

4. Use what you know about inscribed and central angles to explain why the angle inscribed in a semicircle is 90°.

5. Where is the longest chord in any circle? How do you know? Draw a diagram to illustrate your answer.

6. A circle has diameter 16 cm.

a) Which of the following measures could be distances of chords from the centre of this circle? How could you check your answers?

i) 4 cm **ii)** 6 cm **iii)** 8 cm **iv)** 10 cm

b) For each possible distance you identified in part a, determine the length of the chord.

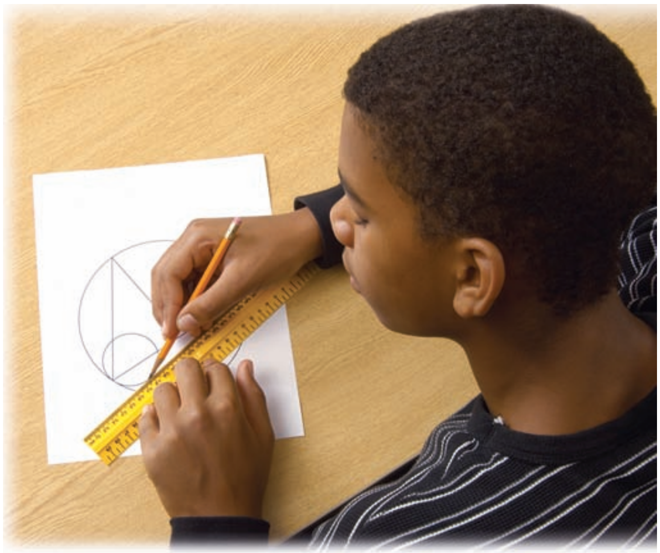
7. a) Construct a circle and mark points P and Q to form major and minor arcs PQ.

b) Construct inscribed ∠PRQ subtended by minor arc PQ.

c) Construct inscribed ∠PSQ subtended by major arc PQ.

d) How are ∠PRQ and ∠PSQ related? Justify your answer.

Unit Problem Circle Designs



Many works of art, designs, and objects in nature are based on circles.
Work with a partner to generate a design for a corporate or team logo.

Part 1
Sketch a design that uses circles, tangents, and chords.
Use your imagination to relate circles to a business or sports team.

Part 2
Work with geometry tools or computer software to draw your design.
Measure and label all angles and lengths that demonstrate the circle properties.
Some lines or features at this stage may disappear or be covered by the final coloured copy. So, ensure you have a detailed design copy to submit that demonstrates your understanding of the geometry.

Part 3
Produce a final copy of your design. You may cover or alter the underlying geometry features at this point if it enhances your design.

Your work should show:
- sketches of your design
- a detailed, labelled copy of your design that shows circle geometry properties
- written explanations of the circle properties you used in your design
- a final coloured copy of your design with an explanation of its purpose, if necessary

Reflect on Your Learning

Explain how knowing the circle properties from this unit can help you determine measurements of lengths and angles in circles.